ECED 3300
Tutorial 5

Problem 1

A point charge Q is located at the origin. Find the electric flux that crosses

(a) a sphere of radius R centered at the origin;

(b) the annulus, described by the condition $\alpha_1 \leq \phi \leq \alpha_2$, which lies on a sphere of radius R centered at the origin;

(c) the patch, specified by $\alpha_1 \leq \phi \leq \alpha_2$ and $\theta_1 \leq \theta \leq \theta_2$, which lies on a sphere of radius R centered at the origin.

Solution

(a) By definition, the flux through the entire sphere is equal to the total charge enclosed by the surface of the sphere, $\Psi = Q$.

(b) By definition, $\Psi = \int_S \mathbf{D} \cdot d\mathbf{S}$. In our case, $d\mathbf{S} = \mathbf{a}_r R^2 \sin\theta d\theta d\phi$, and $\mathbf{D} = Q/(4\pi r^2)\mathbf{a}_r$ (the flux density due to a point charge at the origin). It follows that

$$\Psi = \frac{Q}{4\pi R^2} R^2 \int_0^\pi \sin\theta d\theta \int_{\alpha_1}^{\alpha_2} d\phi = \frac{Q}{4\pi}(\alpha_2 - \alpha_1) \times \cos\theta \Big|_\pi^0 = \frac{Q(\alpha_2 - \alpha_1)}{2\pi}.$$

(c) Similarly to part (b):

$$\Psi = \frac{Q}{4\pi} \int_{\theta_1}^{\theta_2} \sin\theta d\theta \int_{\alpha_1}^{\alpha_2} d\phi = \frac{Q(\cos\theta_1 - \cos\theta_2)(\alpha_2 - \alpha_1)}{4\pi}.$$

Problem 2

The interior of an infinitely long cylinder of radius a is filled with an axially symmetric volume charge distribution, $\rho_v(\rho) = \kappa\frac{\rho}{a}$, (C/m^3), where $\kappa = const$. The charged cylinder is surrounded by an infinitely long, thin cylindrical shell of the radius b, $b > a$, which carries the charge ρ_l, (C/m), per unit length. Assuming the dielectric constants of all media to be equal to unity, find the electric field everywhere.

Solution

By axial symmetry of the system, the electric field can only have a radial component; a cylinder of length L and radius r will be chosen as a special Gaussian surface. We can then apply Gauss's law to the charge distribution enclosed by such a surface. Let us consider the following cases

1. $\rho \le a$. Consider a cylinder of radius ρ, $\rho \le a$ and length L as a Gaussian surface. Applying Gauss's law, we obtain the equation

$$D\,2\pi\rho L = L\int_0^{2\pi} d\phi \int_0^{\rho} \rho' d\rho' \kappa\rho'/a.$$

Performing the elementary integration on the right-hand side, we arrive at

$$D\,2\pi\rho L = 2\pi L\frac{\kappa\rho^3}{3a} \quad \Longrightarrow \quad D = \frac{\kappa\rho^2}{3a}.$$

Due to the relation between the flux density and the electric field, $\mathbf{D} = \epsilon_0\mathbf{E}$, we obtain at once the final expression for the field

$$\mathbf{E} = \frac{\kappa\rho^2}{3\epsilon_0 a}\,\mathbf{a}_\rho.$$

2. $a \le \rho \le b$. By the same token,

$$D\,2\pi\rho L = 2\pi L\int_0^{a} \rho' d\rho' \kappa\rho'/a.$$

It follows that

$$\mathbf{E} = \frac{\kappa a^2}{3\epsilon_0 \rho}\,\mathbf{a}_\rho$$

in this region.

3. $\rho \ge b$. Here, the flux density is determined by the total charge inside the cylinder and the shell:

$$D\,2\pi\rho L = 2\pi L\frac{\kappa a^2}{3} + \rho_l L$$

Thus

$$\mathbf{E} = \frac{\kappa a^2}{3\epsilon_0 \rho}\left(1 + \frac{3\rho_l}{2\pi\kappa a^2}\right)\,\mathbf{a}_\rho.$$

Problem 3

Determine the charge density due to the following electric charge distributions

(a) $\mathbf{D} = 4\rho\sin\phi\mathbf{a}_\rho + 2\rho\cos\phi\mathbf{a}_\phi + 2z^2\mathbf{a}_z, \;\; C/m^2.$

(b) $\mathbf{D} = \frac{2\cos\theta}{r^3}\mathbf{a}_r + \frac{\sin\theta}{r^3}\mathbf{a}_\theta, \;\; C/m^2.$

Solution

Using the differential form of Gauss's law,

$$\nabla \cdot \mathbf{D} = \rho_v.$$

(a) Thus

$$\rho_v = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho D_\rho) + \frac{1}{\rho}\frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z}.$$

It follows that

$$\rho_v = \frac{1}{\rho}\frac{\partial}{\partial \rho}(4\rho^2 \sin\phi) + 2\frac{\partial(\cos\phi)}{\partial \phi} + \frac{\partial(2z^2)}{\partial z} = 6\sin\phi + 4z.$$

(b) By the same token,

$$\rho_v = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, D_\theta) + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial \phi}.$$

It can be inferred that

$$\rho_v = \frac{2\cos\theta}{r^2}\frac{\partial}{\partial r}\left(\frac{1}{r}\right) + \frac{1}{r^4\sin\theta}\frac{\partial}{\partial \theta}(\sin^2\theta) = -\frac{2\cos\theta}{r^4} + \frac{2\cos\theta}{r^4} = 0.$$